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An Algebraic Approach to the Generalized Symmetrical Double-Well Potential

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Abstract We have obtained the energy eigenvalues and the corresponding eigenfunctions for the generalized double-well potential in the non-relativistic Schrödinger equation. We have calculated the creation and annihilation operators directly from the eigenfunction and we have shown these operators satisfy the commutation relation of the $SU(2)$ group. We have expressed the Hamiltonian in terms of the $su(2)$ algebra. Some interesting result including the standard symmetrical double-well potential, reflectionless-type potential and $V_0 \tanh^2(r/d)$ potential are also discussed.

Keywords Double-well potential · Lie algebras · Algebraic approach · Ladder operators

1 Introduction

In resent years, there has been an increasing interest in the study of quantum mechanical problems by the Lie algebraic methods [1–4], because these methods have been the subject of interest in many fields of physics and chemistry. For example these methods provide a way to obtain the wavefunctions of potentials in nuclear [5–7], and polyatomic molecules [8–12]. On the other hand, the deformed algebras are deformed versions of the usual Lie algebras where obtained by introducing a deformation parameter q. The deformed algebras provide appropriate tools for describing systems which cannot be described by the ordinary Lie algebras. We study the dynamical group for generalized symmetrical double-well potential in the Schrödinger equation. The symmetrical double-well potential offered by Büyükkilic et al. [13] to describe the vibrational of polyatomic molecules. They obtained the bound state energy eigenvalue and the corresponding eigenfunctions for this potential by using the Nikiforov-Uvarov method. They also calculated the solutions of the Ammonia molecule (NH3) with the help of this potential. Then, Yang [14] generalized the symmetrical

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double-well potential by using the deformed hyperbolic functions [15–18] and applied the supersymmetry WKB method to obtain the energy eigenvalue of this potential. Also the solutions of this potential in the relativistic Klein-Gordon and Dirac equations have been determined by the Shape invariance method [19]. For the case $q = 1$, we have the same relations for standard double-well potential [20]. We obtain the creation and annihilation operators directly from the eigenfunction. Then we show these operators construct the dynamical algebra $su(2)$, by the factorization methods [21, 22].

This paper is organized as follows. In Sect. 2 we consider one-dimensional Schrödinger equation for generalized symmetrical double-well potential, then we obtain the normalized wave function of this potential. In Sect. 3 we establish the ladder operators for this potential, also we show that these operators construct the dynamical algebra $su(2)$. By choosing appropriate parameters we obtain the ladder operators for standard double-well potential, reflectionless-type potential and $V_0 \tanh^2(x/d)$ in Sect. 4. Finally, the conclusions is given in Sect. 5.

2 The Eigenvalues and Eigenfunctions

The generalized symmetrical double-well potential is given by [14]

$$V(x) = V_1 \tanh_q^2 \alpha x - \frac{V_2}{\cosh_q^2 \alpha x}, \tag{1}$$

where the range of parameter q is $-1 \le q < 0$ or $q > 0$, and the deformed hyperbolic functions are defined as [15–18].

$$\begin{aligned} \sinh_q x &= \frac{e^x - qe^{-x}}{2}, \qquad \cosh_q x = \frac{e^x + qe^{-x}}{2}, \\ \mathrm{cosech}_q x &= \frac{1}{\sinh_q x}, \qquad \mathrm{sech}_q x = \frac{1}{\cosh_q x}, \\ \tanh_q x &= \frac{\sinh_q x}{\cosh_q x}, \qquad \coth_q x = \frac{\cosh_q x}{\sinh_q x}. \end{aligned} \tag{2}$$

The 1-dimensional Schrödinger equation with generalized symmetrical double-well potential is as follows:

$$\left[-\frac{\hbar^2}{2M}\frac{d^2}{dx^2} + V_1 \tanh_q^2 \alpha x - \frac{V_2}{\cosh_q^2 \alpha x}\right]\phi_n(x) = E_n \phi_n(x). \tag{3}$$

Putting $u = \tanh_q \alpha x$, we can obtain

$$\left[\frac{d}{du}(1-u^2)\frac{d}{du} + \nu(\nu+1) - \frac{\mu^2}{1-u^2}\right]\phi_n(x) = 0, \tag{4}$$

where

$$\nu(\nu+1) = \frac{2M}{\alpha^2\hbar^2}\left(V_1 + \frac{V_2}{q}\right), \qquad \mu^2 = \frac{2M}{\alpha^2\hbar^2}(V_1 - E_n). \tag{5}$$

We take the following ansatz for the eigenfunction

$$\phi_n(x) = (1-u^2)^{\frac{\mu}{2}} f(u), \tag{6}$$

therefore we can rewrite (4) as

$$(1-u^2)\frac{d^2 f(u)}{du^2} - 2u(\mu+1)\frac{df(u)}{du} + [\nu(\nu+1) - \mu(1+\mu)]f(u) = 0. \tag{7}$$

Setting $\rho = \frac{1}{2}(1-u)$, the above equation turns to

$$\rho(1-\rho)\frac{d^2 f(\rho)}{d\rho^2} + (1+\mu)(1-2\rho)\frac{df(\rho)}{d\rho} - (\mu-\nu)(1+\nu+\mu)f(\rho) = 0. \tag{8}$$

Comparing (8) with the following hypergeometric function [23]

$$\left[x(1-x)\frac{d^2}{dx^2} + [c-(a+b+1)x]\frac{d}{dx} - ab\right]{}_2F_1(a,\, b,\, c;\, x) = 0, \tag{9}$$

and from the behaviors of the wavefunctions at $\rho = 0, 1, \infty$, the solutions of the (8) can be written as

$$\phi_n(u) = (1-u^2)^{\frac{\mu}{2}}\, {}_2F_1\left(\mu-\nu,\, \mu+\nu+1,\, \mu+1;\, \frac{1-u}{2}\right). \tag{10}$$

Also from consideration of the finiteness of the wavefunction (10) it is shown that the general quantum condition is

$$\mu - \nu = -n, \quad n = 0,\, 1,\, 2, \ldots. \tag{11}$$

The energy eigenvalue can be determined by the constrained condition

$$E_n = V_1 - \frac{\alpha^2\hbar^2}{2M}(\nu-n)^2. \tag{12}$$

We use the following relation between hypergeometric functions and Gegenbauer polynomials:

$$C_n^{\lambda}(x) = \frac{\Gamma(2\lambda+n)}{n!\Gamma(2\lambda)}\, {}_2F_1\left(-n,\, 2\lambda+n,\, \frac{1}{2}+\lambda;\, \frac{1-x}{2}\right). \tag{13}$$

Now we can write the wavefunction as

$$\phi_n(u) = N_n(1-u^2)^{\frac{\mu}{2}}\; C_n^{\nu-n+\frac{1}{2}}(u), \tag{14}$$

where N_n is the normalized factor to be determined below

$$\int_{-\infty}^{\infty} |\phi_n(x)|^2 dx = \frac{|N_n|^2}{\alpha}\int_{-1}^{1}(1-u^2)^{\nu-n-1}\left[C_n^{\nu-n+\frac{1}{2}}(u)\right]^2 du = 1. \tag{15}$$

By using the following relation we can calculate the above integral.

$$\int_{-1}^{1}(1-x^2)^{\beta-\frac{3}{2}}\; [C_n^{\beta}(x)]^2 dx = \frac{\pi^{1/2}\Gamma(\beta-1/2)\Gamma(2\beta+n)}{n!\Gamma(\beta)\Gamma(2\beta)}, \tag{16}$$

we have

$$N_n = \sqrt{\frac{\alpha n!(\nu-n-\frac{1}{2})!(2\nu-2n)!}{\pi^{\frac{1}{2}}(\nu-n-1)!(2\nu-n)!}}. \tag{17}$$

3 The Construction of the Ladder Operators

Now we would like to find the creation and annihilation operators for the wavefunctions with the factorization method. We define the q-deformed ladder operators $\hat{L}_{q\pm}$ with the property

$$\hat{L}_{q\pm}\phi_n^\nu(x) = l_{q\pm}\phi_{n\pm1}^\nu(x). \tag{18}$$

We consider the following ansatz for ladder operators,

$$\hat{L}_{q\pm} = A_\pm(x)\frac{d}{dx} + B_\pm(x). \tag{19}$$

By the action of the differential operator $\frac{d}{d\rho}$ on the wave function (14) we can obtain

$$\frac{d}{du}\phi_n^\nu(u) = -\frac{u(\nu-n)}{1-u^2}\phi_n^\nu(u) + \frac{N_n}{N_{n-1}}\frac{(2\nu-2n+1)}{\sqrt{1-u^2}}\phi_{n-1}^\nu(u), \tag{20}$$

where we have used the relation

$$\frac{dC_n^\lambda(u)}{du} = 2\lambda C_{n-1}^{\lambda+1}(u). \tag{21}$$

By using (17), (20) we have

$$\sqrt{1-u^2}\left[\frac{d}{du} + \frac{u(\nu-n)}{1-u^2}\right]\sqrt{\frac{\nu-n+1}{s-n}}\phi_n^\nu(u) = \sqrt{n(2\nu-n+1)}\phi_{n-1}^\nu(u), \tag{22}$$

therefore we define the annihilation operator $\hat{L}_{q-}$ as

$$\begin{aligned}\hat{L}_{q-} &= \sqrt{1-u^2}\left[\frac{d}{du} + \frac{(\nu-n)u}{1-u^2}\right]\sqrt{\frac{\mu+1}{\mu}}\\ &= \frac{1}{\alpha\sqrt{1-\tanh_q^2\alpha x}}\left[\frac{d}{dx} + \frac{\alpha(\nu-n)\tanh_q\alpha x}{\sqrt{1-\tanh_q^2\alpha x}}\right]\sqrt{\frac{\mu+1}{\mu}},\end{aligned} \tag{23}$$

with the following eigenvalue

$$l_{q-} = \sqrt{n(2\nu-n+1)}. \tag{24}$$

Similarly, one can obtain

$$\frac{d}{du}\phi_n^\nu(u) = \frac{u(\nu-n)}{1-u^2}\phi_n^\nu(u) + \frac{N_n}{N_{n+1}}\frac{(2\nu-n)(n+1)}{(2\nu-2n-1)\sqrt{1-u^2}}\phi_{n+1}^\nu(u), \tag{25}$$

where we have used (21) and the following relation

$$\begin{aligned}&2(\lambda-1)(2\lambda-1)C_n^\lambda(u)\\ &\quad= 4\lambda(\lambda-1)(1-u^2)C_{n-1}^{\lambda+1}(u) + (2\lambda+n-1)(n+1)C_{n+1}^{\lambda-1}(u).\end{aligned} \tag{26}$$

Again, by using (17), (25) we obtain

$$\sqrt{1-u^2}\left[-\frac{d}{du}+\frac{u(\nu-n)}{1-u^2}\right]\sqrt{\frac{\nu-n-1}{\nu-n}}\phi_n^{\nu}(u)=\sqrt{(n+1)(2\nu-n)}\phi_{n+1}^{\nu}(u). \tag{27}$$

So, we can define the creation operator $\hat{L}_{q+}$ as

$$\begin{aligned}\hat{L}_{q+}&=\sqrt{1-u^2}\left[-\frac{d}{du}+\frac{(\nu-n)u}{1-u^2}\right]\sqrt{\frac{\mu-1}{\mu}}\\&=\frac{1}{\alpha\sqrt{1-\tanh_q^2\alpha x}}\left[-\frac{d}{dx}+\frac{\alpha(\nu-n)\tanh_q\alpha x}{\sqrt{1-\tanh_q^2\alpha x}}\right]\sqrt{\frac{\mu-1}{\mu}},\end{aligned} \tag{28}$$

with the following eigenvalue

$$l_{q+}=\sqrt{(n+1)(2\nu-n)}. \tag{29}$$

Now, we determine the algebra associated with the operators $\hat{L}_{q-}$, $\hat{L}_{q+}$. Using (14), (24), (29), we calculate the commutator

$$[\hat{L}_{q+},\hat{L}_{q-}]\phi_n^{\nu}(u)=2(\nu-n)\phi_n^{\nu}(u)=2\mu\phi_n^{\nu}(u). \tag{30}$$

Then, we define the operator $\hat{L}_{q0}$ as follows,

$$\hat{L}_{q0}\phi_n^{\nu}(u)=\mu\phi_n^{\nu}(u), \tag{31}$$

therefore the operators $\hat{L}_{q\pm}$, $\hat{L}_{q0}$ satisfy the following commutation relations

$$[\hat{L}_{q+},\hat{L}_{q-}]=2\hat{L}_{q0},\qquad [\hat{L}_{q0},\hat{L}_{q+}]=+\hat{L}_{q+},\qquad [\hat{L}_{q0},\hat{L}_{q-}]=-\hat{L}_{q-}, \tag{32}$$

which correspond to the $su(2)$ algebra. The Casimir operator can be obtained as

$$\hat{C}=\hat{L}_{q0}(\hat{L}_{q0}-1)+\hat{L}_{q+}\hat{L}_{q-} \tag{33}$$

with the following eigenvalue

$$c=\nu(\nu+1)-2\mu. \tag{34}$$

The Hamiltonian can be written as

$$\hat{H}=V_1-\frac{\alpha^2\hbar^2}{2M}\hat{L}_{q0}^2. \tag{35}$$

4 Discussion

In this section, we obtain the creation and annihilation operators for the standard symmetrical double-well potential, reflectionless-type potential and $V_0\tanh^2(r/d)$ potential by choosing appropriate parameters in the generalized double-well potential model.

4.1 Standard Symmetrical Double-Well Potential

If we choose $q=1$, the generalized symmetrical double-well potential model given in (1) reduces to standard symmetrical double-well potential [20]

$$V(x) = V_1 \tanh^2 \alpha x - \frac{V_2}{\cosh^2 \alpha x}. \tag{36}$$

Substituting the corresponding parameter into (23), (28), we obtain the ordinary ladder operators and corresponding eigenvalues for the standard symmetrical double-well potential as follows

$$\begin{aligned}
\hat{L}_- &= \frac{1}{\alpha\sqrt{1-\tanh^2\alpha x}}\left[\frac{d}{dx} + \frac{\alpha(\nu_1 - n)\tanh\alpha x}{\sqrt{1-\tanh^2\alpha x}}\right]\sqrt{\frac{\mu_1+1}{\mu_1}},\\
l_- &= \sqrt{n(2\nu_1 - n + 1)},\\
\hat{L}_+ &= \frac{1}{\alpha\sqrt{1-\tanh^2\alpha x}}\left[\frac{-d}{dx} + \frac{\alpha(\nu_1 - n)\tanh\alpha x}{\sqrt{1-\tanh^2\alpha x}}\right]\sqrt{\frac{\mu_1-1}{\mu_1}},\\
l_+ &= \sqrt{(n+1)(2\nu_1 - n)},
\end{aligned} \tag{37}$$

where

$$\nu_1(\nu_1+1) = \frac{2M}{\alpha^2\hbar^2}(V_1+V_2), \qquad \mu_1 = \nu_1 - n. \tag{38}$$

One can show these operators satisfy the $su(2)$ algebra

$$[\hat{L}_+, \hat{L}_-] = 2\hat{L}_0, \qquad [\hat{L}_0, \hat{L}_+] = +\hat{L}_+, \qquad [\hat{L}_0, \hat{L}_-] = -\hat{L}_-, \tag{39}$$

with

$$\hat{L}_0\phi_n(x) = \mu_1\phi_n(x). \tag{40}$$

4.2 Reflectionless-Type Potential

If we make the replacement $q=1$, $\alpha=1$, $V_1=0$ and $V_2=\frac{1}{2}\lambda(\lambda+1)$, the generalized double-well potential model given in (1) becomes the reflectionless-type potential [24]

$$V(x) = -\frac{1}{2}\lambda(\lambda+1)\,\mathrm{sech}\, x, \tag{41}$$

where λ is an integer, i.e., $\lambda = 1, 2, 3, \ldots$. Making the corresponding parameter into (23), (28), we obtain the ladder operators for the reflectionless-type potential as

$$\begin{aligned}
\hat{L}_- &= \frac{1}{\sqrt{1-\tanh^2 x}}\left[\frac{d}{dx} + \frac{(\nu_2 - n)\tanh x}{\sqrt{1-\tanh^2 x}}\right]\sqrt{\frac{\mu_2+1}{\mu_2}},\\
l_- &= \sqrt{n(2\nu_2 - n + 1)},\\
\hat{L}_+ &= \frac{1}{\sqrt{1-\tanh^2 x}}\left[-\frac{d}{dx} + \frac{(\nu_2 - n)\tanh x}{\sqrt{1-\tanh^2 x}}\right]\sqrt{\frac{\mu_2-1}{\mu_2}},\\
l_+ &= \sqrt{(\nu_2+1)(2\nu_2 - n)},
\end{aligned} \tag{42}$$

where

$$\nu_2(\nu_2+1) = M\lambda(\lambda+1)/\hbar^2, \qquad \mu_2 = \nu_2 - n. \tag{43}$$

These operators satisfy the $su(2)$ algebra.

$$[\hat{L}_+, \hat{L}_-] = 2\hat{L}_0, \qquad [\hat{L}_0, \hat{L}_+] = +\hat{L}_+, \qquad [\hat{L}_0, \hat{L}_-] = -\hat{L}_-, \tag{44}$$

with

$$\hat{L}_0\phi_n(x) = \mu_2\phi_n(x). \tag{45}$$

4.3 $V_0 \tanh^2(x/d)$ Potential

If we make the replacement $q = 1$, $\alpha = 1/d$, $V_1 = V_0$ and $V_2 = 0$, the generalized double-well potential model given in (1) yields $V_0 \tanh^2(x/d)$ potential [25]

$$V(x) = V_0 \tanh^2(x/d). \tag{46}$$

This potential can be converted to the modified Pöschl-Teller potential with a constant shift [26, 27], i.e., $V(x) = V_0 - V_0/\cosh^2(x/d)$ where Dong et al. [28, 29] have investigated the energy spectra and wave function of this potential by using the ladder operators and $SU(2)$ group methods. Substituting the corresponding parameters into (23), (28), we can consequently obtain the ladder operators for the $V_0 \tanh^2(x/d)$ potential as

$$\begin{aligned}
\hat{L}_- &= \frac{d}{\sqrt{1-\tanh^2(x/d)}}\left[\frac{d}{dx} + \frac{(\nu_3 - n)\tanh(x/d)}{d\sqrt{1-\tanh^2(x/d)}}\right]\sqrt{\frac{\mu_3+1}{\mu_3}},\\
l_- &= \sqrt{n(2\nu_3 - n + 1)},\\
\hat{L}_+ &= \frac{d}{\sqrt{1-\tanh^2(x/d)}}\left[-\frac{d}{dx} + \frac{(\nu_1 - n)\tanh(x/d)}{d\sqrt{1-\tanh^2(x/d)}}\right]\sqrt{\frac{\mu_3-1}{\mu_3}},\\
l_+ &= \sqrt{(n+1)(2\nu_3 - n)},
\end{aligned} \tag{47}$$

where

$$\nu_3(\nu_3+1) = 2Md^2V_0/\hbar^2, \qquad \mu_3 = \nu_3 - n. \tag{48}$$

These operators satisfy the following commutation relations

$$[\hat{L}_+, \hat{L}_-] = 2\hat{L}_0, \qquad [\hat{L}_0, \hat{L}_+] = +\hat{L}_+, \qquad [\hat{L}_0, \hat{L}_-] = -\hat{L}_-, \tag{49}$$

with

$$\hat{L}_0\phi_n(x) = \mu_3\phi_n(x), \tag{50}$$

which correspond to $su(2)$ algebra.

5 Conclusion

In this paper, we have obtained the bound state energy eigenvalue and corresponding eigenfunctions for generalized double-well potential in one-dimensional non-relativistic

Schrödinger equation. We have derived the ladder operators, then we have shown these operators satisfy the $SU(2)$ group. We have expressed the Hamiltonian in terms of the $su(2)$ algebra. For the case $q = 1$, we have calculated these solutions for standard double-well potential. By choosing appropriate parameters we have obtained the same relations for reflectionless-type potential and $V_0 \tanh^2(r/d)$ potential as special cases.

References

1. Setare, M.R., Karimi, E.: Phys. Scr. **75**, 90 (2007)
2. Setare, M.R., Karimi, E.: Int. J. Theor. Phys. **46**, 1381 (2007)
3. Setare, M.R., Olfati, Gh.: Phys. Scr. **75**, 250 (2007)
4. Dong, S.H., Lara-Rosana, F., Sun, G.-H.: Phys. Lett. A **325**, 218 (2004)
5. Arima, A., Iachello, F.: Ann. Phys. (NY) **99**, 253 (1976)
6. Arima, A., Iachello, F.: Ann. Phys. (NY) **111**, 201 (1978)
7. Arima, A., Iachello, F.: Ann. Phys. (NY) **123**, 468 (1979)
8. Von Rosmalen, O.S., Iachello, F., Levine, R.D., Dieperink, A.E.: J. Chem. Phys. **79**, 2515 (1983)
9. Von Rosmalen, O.S., Benjamin, I., Levine, R.D.: J. Chem. Phys. **81**, 5986 (1984)
10. Iachello, F., Oss, S.: Phys. Rev. Lett. **66**, 2976 (1991)
11. Iachello, F., Ibrahim, M.: J. Chem. Phys. A **102**, 9427 (1998)
12. Iachello, F., Levine, R.D.: Algebraic Theory of Molecules. Oxford University Press, New York (1995)
13. Büyükkilic, F., Eğrifes, H., Hemirhan, D.: Theor. Chem. Acc. **98**, 192 (1997)
14. Yang, Q.B.: Acta Photonica Sin. **32**, 882 (2003)
15. Arai, A.: J. Math. Anal. Appl. **158**, 63 (1991)
16. Jia, C.S., Li, S.C., Li, Y., Sun, L.T.: Phys. Lett. A **300**, 115 (2002)
17. Jia, C.S., Li, Y., Sun, Y., Liu, J.Y., Sun, L.T.: Phys. Lett. A **311**, 115 (2003)
18. de Souza Dutra, A.: Phys. Lett. A **339**, 252 (2005)
19. Zhao, X.Q., Jia, C.S., Yang, Q.B.: Phys. Lett. A **337**, 189 (2005)
20. Büyükkilic, F., Eğrifes, H., Hemirhan, D.: Theor. Chem. Acc. **98**, 192 (1997)
21. Dong, S.-H.: Factorization Method in Quantum Mechanics. Springer, Berlin (2007)
22. Dong, S.-H.: Int. J. Quantum Chem. **86**, 433 (2002)
23. Gradshteyn, I.S., Ryzhlik, I.M.: Tables of Integrals, Series and Products, 5th edn. Academic Press, New York (1994)
24. Chen, G., Lou, Z.M.: Acta Phys. Sin. **52**, 1071 (2003) (in Chinese)
25. Qiang, W.C.: Chin. Phys. **13**, 571 (2004)
26. Landau, L.D., Lifshitz, E.M.: Quantum Mechanics, Non-Relativistic Theory, 3rd edn. Pergamon, Oxford (1977)
27. Flügge, S.: Practical Quantum Mechanics. Springer, Berlin (1971)
28. Dong, S.H., Lemus, R.: Int. J. Quantum Chem. **86**, 265 (2002)
29. Dong, S.S., Dong, S.H.: Czech. J. Phys. **52**, 753 (2002)